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Topology and Differentiable Structures of Mapping Space Quotients

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Abstract

In this survey article we show the basic method how to give *topological structures* on mapping spaces or mapping space quotients and moreover introduce a new method to provide *differentiable structures* on them with illustrative examples.

1 Introduction.

To describe complicated phenomena, we need an infinite number of parameters, since our real world has, of course, infinite dimension. However, on the other hand, we are able to understand fully only finite dimensional objects. Therefore we have to pick up a finite number of parameters, according to each purpose, among an infinite number of parameters. We extract finite dimensional objects from infinite dimensional objects and naturally such parameters enjoy several relations or conditions. Thus we proceed to study finite dimensional *manifolds.* In this stage, geometrical study plays an important role. Then, in the study of finite dimensional manifolds, functions on manifolds and mappings between manifolds are investigated. The space of mappings are of infinite dimensional. Then, again we extract finite dimensional objects from infinite dimensional mapping spaces. Thus our understanding is improved, step by step, on various complicated phenomena and the mathematical structures behind them as well. Now we are going to start with finite dimensional manifolds.

Let N, M be differentiable manifolds of dimension n, m respectively. In this article we only treat C^∞ i.e. infinitely differentiable manifolds and C^∞ mappings, otherwise stated.

Let $C^\infty(N, M)$ denote the set of differentiable mappings from N to M, and X be a subset of $C^\infty(N, M)$. Suppose an equivalence relation $\sim$ is given on X. Then $X/\sim$ denotes the quotient space (*mapping space quotient*). The problem is how to give a topological structure and a differentiable structure on the mapping space quotient $X/\sim$.

Example 1.1 (The space of knots.) An embedding from the circle S^1 to the space $\mathbf{R}^3$ or its image is called a *knot.* Let $\mathrm{Emb}(S^1, \mathbf{R}^3) \subset C^\infty(S^1, \mathbf{R}^3)$ denote the set consisting of knots (the space of knots). Then the *knot theory* treats connected components of the space of knots $\mathrm{Emb}(S^1, \mathbf{R}^3)$. Moreover we can define on it various geometric structures (e.g. a symplectic structure, a complex structure and so on. See [8]. See also subsection 6.3 of this article.

Example 1.2 (The diffeomorphism group.) The space $\mathrm{Diff}(N)$ of diffeomorphisms on N can be endowed with the structure of a topological group and an infinite dimensional Lie group. For example the famous theorem stating that the space $\mathrm{Diff}^+(S^2)$ of orientation-preserving diffeomorphisms on the sphere S^2 is homotopy equivalent to the special orthogonal group $\mathrm{SO}(3)$ need the topology on the mapping space $\mathrm{Diff}^+(S^2)$.

Example 1.3 (The super space of Riemannian structures.) Let $\mathcal{R}_N$ be the set of Riemannian structures on N. Then $\mathcal{R}_N$ is regarded as a mapping space. The diffeomorphism group $\mathrm{Diff}(N)$ naturally acts on $\mathcal{R}_N$ and the orbit space or the quotient space $\mathcal{R}_N/\mathrm{Diff}(N)$ which is designated by $\mathcal{S}_N$ is the space of isomorphisms classes of Riemannian structures on N, which is called the super space. See subsection 6.4.

Example 1.4 (The variational method.) Let $\Phi = \Phi(f)$ be a real valued function on the mapping space $C^\infty(N, M)$. Then the variable f means a differentiable mapping from N to M. A mapping $f \in C^\infty(N, M)$ is called a *critical point* of Φ if, for any one-parameter deformation f_t, $\frac{d}{dt}\Phi(f_t)|_{t=0} = 0$. Based on this idea, later we will give a differentiable structure on $C^\infty(N, M)$. See section 5.

Example 1.5 (Stability and the classification problem of mappings.) On the mapping space $C^\infty(N, M)$, the product group $\mathrm{Diff}(N) \times \mathrm{Diff}(M)$ of diffeomorphism groups naturally acts. Then we call a mapping $f \in C^\infty(N, M)$ is *stable* if the orbit through f forms an open subset of $C^\infty(N, M)$, namely if any f' belonging to some neighbourhood of f is transformed to f via the action of $\mathrm{Diff}(N) \times \mathrm{Diff}(M)$. The purpose of the *differential topology of mapping* is to study on the the quotient space $\mathcal{M} := C^\infty(N, M)/\mathrm{Diff}(N) \times \mathrm{Diff}(M)$. Considering, for each point $x_0 \in N$, the germ of $f \in C^\infty(N, M)$ at x_0, we define the equivalence relation $\sim_{x_0}$ on the set $C^\infty(N, M)$. The quotient space $C^\infty(N, M)/\sim_{x_0}$ represents the space of germs $f : (N, x_0) \to M$ of differentiable mappings. We will give the topology and the differentiable structure on $C^\infty(N, M)/\sim_{x_0}$. Then the purpose of the *singularity theory of mappings* is to study on the various (further) quotient spaces of $C^\infty(N, M)/\sim_{x_0}$.

First we recall how to introduce several topologies on mapping spaces in §2 for the case of Cartesian spaces, and in §3 for the case of finite dimensional manifolds. In §4 we treat the space of map-germs. We explain our main theory of differentiable structures on mapping space quotients in §5, and provide several examples and applications of the theory in §6.

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2 Topology of mapping spaces between Cartesian spaces

We denote by $C^0(\mathbf{R}^n, \mathbf{R}^m)$ the set of C^0 (i.e. continuous) mappings from $\mathbf{R}^n$ to $\mathbf{R}^m$.

Then we define the C^0-topology (or compact open topology) on $C^0(\mathbf{R}^n, \mathbf{R}^m)$ by giving its generator: For a compact set $K \subset \mathbf{R}^n$ and an open set $U \subset \mathbf{R}^n \times \mathbf{R}^m$ we set

$$W(K, U) := \{f \in C^0(\mathbf{R}^n, \mathbf{R}^m) \mid j^0 f(K) \subseteq U\},$$

where $j^0 f : \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^m$ is the "graph mapping" defined by $j^0 f(x) = (x, f(x))$. Then $W(K, U)$ is the set of continuous mapping such that the graph over the the given compact set is included in the given open subset of $\mathbf{R}^n \times \mathbf{R}^m$. We take as the generator of the topology the family of subsets of $C^\infty(\mathbf{R}^n, \mathbf{R}^m)$:

$$\{W(K, U) \mid K \subset \mathbf{R}^n \text{ compact},\ U \subset \mathbf{R}^n \times \mathbf{R}^m \text{ open}\}.$$

Thus, for the C^0 topology, a subset $\Omega \subseteq C^0(\mathbf{R}^n, \mathbf{R}^m)$ is an open subset if and only if for any $f \in \Omega$ there exist compact set $K_1, \ldots, K_s$, in $\mathbf{R}^n$ and open subsets $U_1, \ldots, U_s$ in $\mathbf{R}^n \times \mathbf{R}^m$, such that

$$f \in W(K_1, U_1) \cap W(K_2, U_2) \cap \cdots \cap W(K_s, U_s) \subseteq \Omega.$$

Note that if $f \in W(K_1, U_1) \cap W(K_2, U_2)$, then there exist a compact set $K \subset \mathbf{R}^n$ and an open subset $U \subseteq \mathbf{R}^n \times \mathbf{R}^m$ with $f \in W(K, U) \subseteq W(K_1, U_1) \cap W(K_2, U_2)$. In fact it suffices to set $K = K_1 \cup K_2, U = (\pi_1^{-1}(K_1 \cap K_2) \cap U_1 \cap U_2) \cup (U_1 \setminus \pi^{-1}(K_1 \cap K_2)) \cup (U_2 \setminus \pi^{-1}(K_1 \cap K_2))$.

The topological space $C^0(\mathbf{R}^n, \mathbf{R}^m)$ is a Hausdorff space with respect to C^0-topology. Namely, for two mappings $f, g \in C^0(\mathbf{R}^n, \mathbf{R}^m)$, $f \neq g$, there exist an open neighbourhood W of f and an open neighbourhood W' of g satisfying $W \cap W' = \emptyset$.

For a compact set $L \subset \mathbf{R}^n$ and an open subset $V \subseteq \mathbf{R}^m$, we set

$$W'(L, V) := \{f \in C^0(\mathbf{R}^n, \mathbf{R}^m) \mid f(L) \subseteq V\},$$

and consider the topology on $C^0(\mathbf{R}^n, \mathbf{R}^m)$ generated by $\{W'(L, V) \mid L \subset \mathbf{R}^n \text{ compact}, V \subseteq \mathbf{R}^m \text{ open}\}$. Then this topology coincides with the C^0-topology. Therefore the C^0-topology can be said as "the topology of uniform convergence on compact subsets".

The product space $C^0(\mathbf{R}^n, \mathbf{R}^m) \times C^0(\mathbf{R}^n, \mathbf{R}^\ell)$ is homeomorphic to $C^0(\mathbf{R}^n, \mathbf{R}^m \times \mathbf{R}^\ell)$ with respect to C^0-topology.

For a positive integer $r > 0$, we denote by $C^r(\mathbf{R}^n, \mathbf{R}^m)$ the set of C^r-mappings. Then we can induce a topology on the subset $C^r(\mathbf{R}^n, \mathbf{R}^m) \subset C^0(\mathbf{R}^n, \mathbf{R}^m)$ form the C^0-topology on $C^0(\mathbf{R}^n, \mathbf{R}^m)$. We call it the C^0-topology on $C^r(\mathbf{R}^n, \mathbf{R}^m)$.

Next we define the C^1-topology on $C^1(\mathbf{R}^n, \mathbf{R}^m)$.

Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a C^1-mapping. Set $f = (f_1, f_2, \ldots, f_m)$, where $f_i = f_i(x_1, \ldots, x_n) : \mathbf{R}^n \to \mathbf{R}$ are of class C^1. Consider, by the partial derivatives $\dfrac{\partial f_i}{\partial x_j} : \mathbf{R}^n \to \mathbf{R}, (1 \leq i \leq m, 1 \leq j \leq n)$, the 1-jet extension of f:

$$j^1 f : \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{nm} = \mathbf{R}^{n+m+nm}$$

defined by $j^1 f(x) = (x, f(x), \dfrac{\partial f_i}{\partial x_j}(x))$. The mapping $j^1 f$ is obviously continuous.

For a compact set $K \subset \mathbf{R}^n$ and an open subset $U \subseteq \mathbf{R}^{n+m+nm}$, we set

$$W(K, U) := \{f \in C^1(\mathbf{R}^n, \mathbf{R}^m) \mid j^1 f(K) \subseteq U\}$$

The family of subsets

$$\{W(K, U) \mid K \subset \mathbf{R}^n \text{ compact}, U \subseteq \mathbf{R}^{n+nm} \text{ open}\}$$

of $C^1(\mathbf{R}^n, \mathbf{R}^m)$ generate a topology, which is called the C^1-topology on $C^1(\mathbf{R}^n, \mathbf{R}^m)$. The C^1-topology on $C^1(\mathbf{R}, \mathbf{R})$ is stronger than C^0-topology.

As if the C^0-topology is the topology of uniform convergence on compact subsets, the C^1-topology is the topology of uniform convergence together with first derivatives on compact subsets.

Similarly we define the C^2 topology on $C^2(\mathbf{R}^n, \mathbf{R}^m)$. For a C^2-mapping $f \in C^2(\mathbf{R}^n, \mathbf{R}^m)$, we define the 2-jet extension

$$j^2 f : \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{nm} \times \mathbf{R}^{\frac{n(n+1)}{2} m}$$

by $j^2 f(x) = (x, f(x), \frac{\partial f_i}{\partial x_j}(x), \frac{\partial^2 f_i}{\partial x_j \partial x_k}(x))$. For a compact set $K \subset \mathbf{R}^n$ and an open subset U in $\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{nm} \times \mathbf{R}^{\frac{n(n+1)}{2}m}$, we set

$$W(K, U) := \{f \in C^2(\mathbf{R}^n, \mathbf{R}^m) \mid j^2 f(K) \subseteq U\},$$

and consider the topology generated by the family $\{W(K,U)\}$ consisting of such subsets in $C^2(\mathbf{R}^n, \mathbf{R}^m)$. We call it the C^2-topology on $C^2(\mathbf{R}^n, \mathbf{R}^m)$.

Now we introduce the jet space $J^r(\mathbf{R}^n, \mathbf{R}^m)$, for each $r \geq 0$. Motivating on the space of Taylor polynomials, we set

$$J^r(\mathbf{R}^n, \mathbf{R}^m) = \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{nm} \times \mathbf{R}^{\frac{n(n+1)}{2}m} \times \cdots \times \mathbf{R}^N = \mathbf{R}^M,$$

where $N = \left(\begin{array}{c} n+r-1 \\ r \end{array} \right) m$, and

$$M = n + m + nm + \frac{n(n+1)}{2}m + \cdots + \left(\begin{array}{c} n+r-1 \\ r \end{array} \right) m = n + \left(\begin{array}{c} n+r \\ r \end{array} \right) m.$$

Then, for a C^r-mapping $f \in C^r(\mathbf{R}^n, \mathbf{R}^m)$, we define r-jet extension $j^r f : \mathbf{R}^n \to J^r(\mathbf{R}^n, \mathbf{R}^m)$ by

$$j^r f(x) := \left(x, f(x), \frac{\partial f_i}{\partial x_j}(x), \frac{\partial^2 f_i}{\partial x_k \partial_\ell}(x), \ldots, \frac{\partial^r f_i}{\partial x_{j_1} \cdots \partial x_{j_r}}(x) \right).$$

Note that, for $r \geq s$,

$$J^r(\mathbf{R}^n, \mathbf{R}^m) = \{j^r f(x_0) \mid x_0 \in \mathbf{R}^n, f \in C^s(\mathbf{R}^n, \mathbf{R}^m)\}.$$

Then the C^r topology on $C^r(\mathbf{R}^n, \mathbf{R}^m)$ is defined as the topology generated by the family $\{W(K,U)\}$ of subsets

$$W(K, U) := \{f \in C^r(\mathbf{R}^n, \mathbf{R}^m) \mid j^r f(K) \subseteq U\}$$

of $C^r(\mathbf{R}^n, \mathbf{R}^m)$.

Moreover the C^r topology on $C^s(\mathbf{R}^n, \mathbf{R}^m)$ $(s = r, r+1, \ldots, \infty, \omega)$ is induced, since $C^s(\mathbf{R}^n, \mathbf{R}^m) \subseteq C^r(\mathbf{R}^n, \mathbf{R}^m)$.

Lastly we define the C^∞-topology on $C^\infty(\mathbf{R}^n, \mathbf{R}^m)$. For $r \geq 0$, a compact set $K \subset \mathbf{R}^n$ and an open subset $U \subseteq J^r(\mathbf{R}^n, \mathbf{R}^m)$, set

$$W(r, K, U) := \{f \in C^\infty(\mathbf{R}^n, \mathbf{R}^m) \mid j^r f(K) \subseteq U\}.$$

Then the C^∞ topology is defined as the topology generated by $\{W(r,K,U)\}$.

On the space $X = C^\infty(\mathbf{R}^n, \mathbf{R}^m)$ we have a sequence of topologies

$$\mathcal{O}_X^0 \subseteq \mathcal{O}_X^1 \subseteq \mathcal{O}_X^2 \cdots \subseteq \cup_{r=0}^\infty \mathcal{O}_X^r = \mathcal{O}_X^\infty.$$

namely the C^0-topology, the C^1-topology, the C^2-topology, ..., and the C^∞-topology.

Proposition 2.1 *For $0 \le r \le s$, including the case $s = \infty$, the composition of mappings*

$$\Phi : C^s(\mathbf{R}^n, \mathbf{R}^m) \times C^s(\mathbf{R}^m, \mathbf{R}^\ell) \to C^s(\mathbf{R}^n, \mathbf{R}^\ell),\ \Phi(f, g) = g \circ f,$$

is a continuous mapping on the C^r-topology.

Proof: We set $\Phi(f_0, g_0) = g_0 \circ f_0 = h_0 : \mathbf{R}^n \to \mathbf{R}^\ell$. Suppose $h_0 \in W(r, K, U)$ for a compact $K \subset \mathbf{R}^n$ and an open $U \subseteq J^r(\mathbf{R}^n, \mathbf{R}^\ell)$, namely, $j^r h_0(K) \subseteq U$. We set

$$\begin{aligned} & J^r(\mathbf{R}^n, \mathbf{R}^m) \times_{\mathbf{R}^m} J^r(\mathbf{R}^m, \mathbf{R}^\ell) \\ & \qquad = \{(j^r f(x_0), j^r g(y_0)) \in J^r(\mathbf{R}^n, \mathbf{R}^m) \times J^r(\mathbf{R}^m, \mathbf{R}^\ell) \ \mid\ f(x_0) = y_0\} \end{aligned}$$

and define $\varphi : J^r(\mathbf{R}^n, \mathbf{R}^m) \times_{\mathbf{R}^m} J^r(\mathbf{R}^m, \mathbf{R}^\ell) \to J^r(\mathbf{R}^n, \mathbf{R}^\ell)$ by $\varphi(j^r f(x_0), j^r g(y_0)) = j^r(g \circ f)(x_0)$. Then φ is a continuous mapping which is expressed by a polynomial. In general, for $A \subset J^r(\mathbf{R}^n, \mathbf{R}^m), B \subset J^r(\mathbf{R}^m, \mathbf{R}^\ell)$, set

$$A \times_{\mathbf{R}^m} B := \{j^r f(x_0), j^r g(y_0) \in A \times B \ \mid\ f(x_0) = y_0\} \subseteq J^r(\mathbf{R}^n, \mathbf{R}^m) \times_{\mathbf{R}^m} J^r(\mathbf{R}^m, \mathbf{R}^\ell).$$

Then, from the assumption, we have $\varphi((j^r f_0)(K) \times_{\mathbf{R}^m} (j^r g_0)(f_0(K))) \subseteq U$. By Proposition 2.2 below, there exist an open neighbourhood V of $(j^r f_0)(K)$ in $J^r(\mathbf{R}^n, \mathbf{R}^m)$, and an open neighbourhood V' of $(j^r g_0)(f_0(K)))$ in $J^r(\mathbf{R}^m, \mathbf{R}^\ell)$ such that $V \times_{\mathbf{R}^m} V' \subseteq \varphi^{-1} U$. Thus we have $\Phi(W(K, V), W(f_0(K), V')) \subseteq W(K, U)$, and we see that Φ is continuous. □

Proposition 2.2 ([21][15]) *Suppose A, B, P are Hausdorff spaces, and P locally compact and paracompact, e.g. A, B, P are manifolds. Suppose $\pi : A \to P, \pi' : B \to P$ are continuous mappings, $K \subseteq A, L \subseteq B$ subsets, $\pi|_K : K \to P$, $\pi'|_L : L \to P$ proper. Suppose U' is an open neighbourhood of $K \times_P L = \{(a, b) \in K \times L \mid \pi(a) = \pi(b)\}$ in $A \times_P B = \{(a, b) \in A \times B \mid \pi(a) = \pi'(b)\}$. Then there exist an open neighbourhood V of K in A and an open neighbourhood V' of L in B such that $V \times_P V' \subseteq U'$.*

Remark 2.3 Remark that the C^r-topology ($r = 0, 1, 2, \ldots, \infty$) is generated by a countable number of subsets. In fact,

$$\{W(r, \overline{U_{1/k}(a)}, U_{1/\ell}(b)) \mid a \in \mathbf{Q}^n, b \in \mathbf{Q}^M, k = 1, 2, \ldots, \ell = 1, 2, \ldots\}$$

generates the C^r-topology, where $\mathbf{Q}^M \subset \mathbf{R}^M = J^r(\mathbf{R}^n, \mathbf{R}^m)$ is the set of rational points.

We define other topologies on $C^\infty(\mathbf{R}^n, \mathbf{R}^m)$ defined by H. Whitney: For an open subset $U \subseteq \mathbf{R}^n \times \mathbf{R}^m$, we set

$$W(U) := \{f \in C^\infty(\mathbf{R}^n, \mathbf{R}^m) \mid j^0 f(\mathbf{R}^n) \subseteq U\}.$$

Then the family $\{W(U)\}$ of subsets generates a topology on $C^\infty(\mathbf{R}^n, \mathbf{R}^m)$, which is called the *Whitney C^0-topology*. Also we define the *Whitney C^∞-topology* on $C^\infty(\mathbf{R}^n, \mathbf{R}^m)$, as the topology generated by the family $\{W(r, U) \mid r \ge 0, U \subseteq J^r(\mathbf{R}^n, \mathbf{R}^m)\}$, where

$$W(r, U) = \{f \in C^\infty(\mathbf{R}^n, \mathbf{R}^m) \mid j^r f(\mathbf{R}^n) \subseteq U\}$$

for non-negative integer r and an open subset $U \subseteq J^r(\mathbf{R}^n, \mathbf{R}^m)$, and $j^r f : \mathbf{R}^n \to J^r(\mathbf{R}^n, \mathbf{R}^m)$ is the r-jet extension of f. Similarly we define the *Whitney C^r-topology* on $C^s(\mathbf{R}^n, \mathbf{R}^m)$ $(s \ge r)$.

Set

$$C^s_{pr}(\mathbf{R}^n, \mathbf{R}^m) := \{f \in C^s(\mathbf{R}^n, \mathbf{R}^m) \mid f \text{ is a proper mapping}\}.$$

We can show the following similarly as Proposition 2.1.

Proposition 2.4 *Let $0 \leq r \leq s$, (including the case $s = \infty$). Then the composition*

$$\Phi : C^s_{pr}(\mathbf{R}^n, \mathbf{R}^m) \times C^s(\mathbf{R}^m, \mathbf{R}^\ell) \to C^s(\mathbf{R}^n, \mathbf{R}^\ell), \ \Phi(f, g) = g \circ f,$$

is a continuous mapping with respect to the Whitney C^r-topology.

3 Topology of mapping spaces between manifolds

A *differential manifold* is a space such that, we can take, near each point, a system of coordinates (a local chart) and coordinate transformations are differentiable, for any pair of local charts. The *dimension* of a manifold is defined as the number of coordinates.

Let N be an n-dimensional differentiable manifold, and M an m-dimension differentiable manifold. A mapping $f : N \to M$ is called a *differentiable mapping* if f is continuous and, for any pair of local charts $(U, \varphi), (V, \psi)$, the local representation

$$\boldsymbol{f} = \psi^{-1} \circ f \circ \varphi : \varphi^{-1}(\varphi(U) \cap f^{-1}\psi(V)) \to V$$

illustrated by

$$\begin{array}{ccc} N & \xrightarrow{f} & M \\ \varphi \uparrow & & \uparrow \psi \\ \mathbf{R}^n \supseteq U & & V \subseteq \mathbf{R}^m \end{array}$$

is differentiable.

Now we set

$$C^\infty(N, M) := \{f : N \to M \ \mid f \text{ is a differentiable mapping}\}.$$

Example 3.1 $S^1 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$ is a 1-dimensional differentiable manifold and $\mathbf{R}^2$ is a 2-dimensional differentiable manifold. Then $C^\infty(S^1, \mathbf{R}^2)$ is the set of differentiable closed curves on the plane.

Example 3.2 The space consisting of one point $N = \{\mathrm{pt}\}$ is a 0-dimensional differentiable manifold. $C^\infty(\{\mathrm{pt}\}, M)$ is identified with M by the identification $(f : \mathrm{pt} \to M) \mapsto f(\mathrm{pt}) \in M$.

A mapping $\varphi : N \to N'$ is called a *diffeomorphism* if φ is differentiable, bijective and the inverse mapping φ^{-1} is differentiable.

We introduce the notion of jet space $J^r(N, M)$. For each $x_0 \in N$, we say that two mappings $f : N \to M$ and $g : N \to M$ have the same r-jet at x_0, and write as $f \sim_{r, x_0} g$, if, for a common local chart, the local representations

$$\boldsymbol{f} = (\boldsymbol{f}_1(x_1, \ldots, x_n), \ldots, \boldsymbol{f}_m(x_1, \ldots, x_n)), \quad \boldsymbol{g} = (\boldsymbol{g}_1(x_1, \ldots, x_n), \ldots, \boldsymbol{g}_m(x_1, \ldots, x_n)),$$

have the same partial derivatives at x_0 up to order r, i.e.

$$\frac{\partial^{|\alpha|} \boldsymbol{f}_i}{\partial x^\alpha}(x_0) = \frac{\partial^{|\alpha|} \boldsymbol{g}_i}{\partial x^\alpha}(x_0), \quad 0 \leq |\alpha| \leq r, 1 \leq i \leq m.$$

The equivalence class of f for $\sim_{r,x_0}$ is denoted by $j^r f(x_0)$. Then we define the r-jet space on $N \times M$ by
$$J^r(N,M) = \{j^r f(x_0) \mid x_0 \in N, f \in C^\infty(N,M)\}.$$
Then $J^r(N,M)$ is a differentiable manifold as $J^r(\mathbf{R}^n,\mathbf{R}^m)$ and we have
$\dim J^r(N,M) = \dim J^r(\mathbf{R}^n,\mathbf{R}^m)$.

For $f \in C^r(N,M)$, we define the r-jet extension $j^r f : N \to J^r(N,M)$ by $j^r f(x) = j^r f(x)$. Then $j^r f$ is a differentiable mapping.

Then we introduce the C^∞-*topology* on $C^\infty(N,M)$ as the topology generated by
$$\{W(r,K,U) \mid \; r \geq 0 \text{ integer}, K \subseteq N \text{ compact}, U \subseteq J^r(N,M) \text{ open}\},$$
where, for a non-negative integer r, a compact subset $K \subseteq N$, and for an open subset $U \subseteq J^r(N,M)$ we set
$$W(r,K,U) := \{f \in C^\infty(N,M) \mid j^r f(K) \subseteq U\}.$$
Moreover we introduce *Whitney* C^∞-*topology* on $C^\infty(N,M)$ as the topology generated by $\{W(r,U)\}$ where, for an open subset $U \subseteq J^r(N,M)$,
$$W(r,U) := \{f \in C^\infty(N,M) \mid j^r f(N) \subseteq U\}.$$

Let $f \in C^\infty(N,M)$. If $f \in W(r,U)$ then clearly $f \in W(r,U) \subseteq W(r,K,U)$ for any compact $K \subseteq N$. Therefore C^∞-topology is weaker than Whitney C^∞-topology. If N is compact, then the C^∞ topology and the Whitney C^∞ topology coincide.

For $0 \leq r \leq s \leq \infty$, similarly we define the C^r-topology on the set $C^s(N,M)$ of C^s-mappings $f : N \to M$. Then we have

Proposition 3.3 *Let N, M, L be differentiable manifolds and $0 \leq r \leq s \leq \infty$. The composition*
$$\Phi : C^s(N,M) \times C^s(M,L) \to C^s(N,L), \;\; \Phi(f,g) = g \circ f,$$
is continuous with respect to C^r-topology.

If we set
$$C^s_{pr}(N,M) := \{f \in C^s(N,M) \mid f \text{ is proper}\}.$$

Proposition 3.4 Let N, M, L be differentiable manifolds and $0 \leq r \leq s \leqq \infty$. The composition
$$\Phi : C^s_{pr}(N,M) \times C^s(M,L) \to C^s(N,L), \;\; \Phi(f,g) = g \circ f,$$
is continuous with respect to C^r-topology.

The proof of Propositions 3.3, 3.4 is established by the same proof as that of Proposition 2.1.

4 Spaces of map-germs

Let $x_0 \in N$. We say that $f, g \in C^\infty(N, M)$ has the same germ at x_0 and write as $f \sim_{x_0} g$, if there exists an open neighbourhood $U \subseteq N$ of x_0 such that $f(x) = g(x)(x \in U)$. The equivalence class of f is denoted by f_{x_0} and called the germ of the mapping f at x_0. The amount of local data of a mapping is contained in its germ completely.

Also for open neighbourhoods Ω and Ω' of x_0 in N and $f \in C^\infty(\Omega, M), g \in C^\infty(\Omega', M)$ we define the relation that f and g have the same germ at x_0 similarly.

Let $x_0 \in N$, Ω an open neighbourhood of x_0 in N, and $f \in C^\infty(\Omega, M)$. Then there exists an $F \in C^\infty(N, M)$ such that f and F have same germ at x_0.

The notation of the germ f_{x_0} for an $f \in C^\infty(N, M)$ and an $x_0 \in N$ is often written $f : (N, x_0) \to (M, y_0)$, where $y_0 = f(x_0)$. For example, a diffeomorphism-germ $\sigma : (\mathbf{R}, x_0) \to (\mathbf{R}, x'_0)$ means the germ of a diffeomorphism $\sigma : \mathbf{R} \to \mathbf{R}$ $(\in C^\infty(\mathbf{R}, \mathbf{R}))$ at $x_0 \in \mathbf{R}$ with $x'_0 = \sigma(x_0)$.

Let $f : (N, x_0) \to (M, y_0)$ and $g : (M, y_0) \to (L, z_0)$ be differentiable map-germs. Then the composition $g \circ f : (N, x_0) \to (L, z_0)$ is well-defined as a differentiable map-germ.

Now we give on the product space $C^\infty(N, M) \times N$ the product topology of the C^∞-topology on $C^\infty(N, M)$ and the manifold topology N. Moreover define an equivalence relation $\sim$ on $C^\infty(N, M) \times N$ by setting $(f, x_0) \sim (g, x'_0)$ if $x_0 = x'_0, f_{x_0} = g_{x_0}$. Consider the quotient space

$$\mathcal{G}(N, M) := (C^\infty(N, M) \times N)/\!\sim,$$

endowed with the quotient topology from the topological space $C^\infty(N, M) \times N$. We call it the *space of map-germs.* Then there are natural continuous mappings $\pi : \mathcal{G}(N, M) \to J^r(N, M)$ defined by $\pi(f_{x_0}) = j^r f(x_0)$, and $\Pi : J^r(N, M) \to N \times M$ defined by $j^r f(x_0) \mapsto (x_0, f(x_0))$.

Two map-germs $f : (N, x_0) \to (M, y_0)$ and $f' : (N', x'_0) \to (M', y'_0)$ are called *diffeomorphic* (or *$\mathcal{A}$-equivalent, right-left equivalent*) and written as $f_{x_0} \sim_{\text{diff}} g_{x'_0}$ if there exist diffeomorphism-germs $\sigma : (N, x_0) \to (N', x'_0)$ and $\tau : (M, y_0) \to (M', y'_0)$ such that $\tau \circ f = f' \circ \sigma : (N, x_0) \to (M', y'_0)$.

Set

$$\Sigma_\infty = \{f_{x_0} \in \mathcal{G}(\mathbf{R}, \mathbf{R}) \mid \frac{d^r f}{dx^r}(x_0) = 0, r = 1, 2, 3, \ldots.\},$$

the set of map-germs with constant Taylor series (flat map-germs). Then we have the following classification theorem:

Theorem 4.1 *The space of diffeomorphism classes* $(\mathcal{G}(\mathbf{R}, \mathbf{R}) \setminus \Sigma_\infty)/\!\sim_{\mathcal{A}}$ *of non-flat map-germs, endowed with the quotient topology is homeomorphic to the space* $\mathbf{N} = \{0, 1, 2, \ldots, \}$ *of natural numbers.*

Here we do not give on $\mathbf{N}$ the discrete topology but we give on it the topology

$$\mathcal{O}_{\mathbf{N}} = \{\{0, 1, \ldots, n\} \mid n \in \mathbf{N}\} \cup \{\emptyset, \mathbf{N}\}$$

induced from the natural ordering of $\mathbf{N}$. The topological space $(\mathbf{N}, \mathcal{O}_{\mathbf{N}})$ is illustrated as $\bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \cdots\cdots.$

Proof of Theorem 4.1. For $g : (\mathbf{R}, x_0) \to (\mathbf{R}, y_0)$, we set

$$\mathrm{ord}_{x_0} g := \min\{r \in \mathbf{N} \mid \frac{d^r(g - g(x_0))}{dx^r}(x_0) \neq 0\}$$

and call it the order of g at x_0. The mapping $\varphi : \mathcal{G}(\mathbf{R},\mathbf{R})\setminus\Sigma_\infty \to \mathbf{N}$ defined by $f_{x_0} \mapsto \mathrm{ord}_{x_0}\dfrac{df}{dx}$ is surjective and continuous. In fact, let $\pi : C^\infty(\mathbf{R},\mathbf{R}) \times \mathbf{R} \to \mathcal{G}(\mathbf{R},\mathbf{R})$ be the natural projection, $(f,x_0) \in (C^\infty(\mathbf{R},\mathbf{R}) \times \mathbf{R}) \setminus \pi^{-1}(\Sigma_\infty)$, and $\mathrm{ord}_{x_0}\dfrac{df}{dx} = n$. Then, if we take $\varepsilon > 0$ sufficiently small, and set $K = [x_0 - \varepsilon, x_0 + \varepsilon]$, $V = (x_0 - \varepsilon, x_0 + \varepsilon)$,

$$U = \{j^{n+1}g(x) \in J^n(\mathbf{R},\mathbf{R}) \mid |g^{(n+1)}(x) - f^{(n+1)}(x_0)| < \varepsilon\},$$

then $(g,x) \in W(n,K,U) \times V$ implies $\mathrm{ord}_x\dfrac{dg}{dx} \leq n$. Moreover φ is an open mapping, i.e. φ maps any open subset to an open subset. In fact, for any open neighbourhood W of (f,x_0) and any integer ℓ with $0 \leq \ell \leq \varphi(f_{x_0}) = \mathrm{ord}_{x_0}\dfrac{df}{dx}$, we set $g(x) = f(x) + \varepsilon(x)(x-x_0)^{\ell+1}$, where ε is a differentiable function satisfying that $\varepsilon(x_0) \neq 0$ and vanishing outside of a neighbourhood of x_0. Then we choose ε appropriate, then $g \in W$ and $\mathrm{ord}_{x_0}\dfrac{dg}{dx} = \ell$. The mapping φ induces $\bar\varphi : (\mathcal{G}(\mathbf{R},\mathbf{R}) \setminus \Sigma_\infty)/{\sim_\mathcal{A}} \to \mathbf{N}$. Then $\bar\varphi$ is a bijection. In fact, if $\mathrm{ord}_{x_0}\dfrac{df}{dx} = n$ then f_{x_0} is diffeomorphic to the germ of x^{n+1} at 0. Moreover φ is continuous and an open mapping. Since $\bar\varphi^{-1} : \mathbf{N} \to (\mathcal{G}(\mathbf{R},\mathbf{R}) \setminus \Sigma_\infty)/{\sim_\mathcal{A}}$ is also continuous, we see $\bar\varphi$ is a homeomorphism. □

Next we consider

$$\mathcal{G}(\mathbf{R}^2,\mathbf{R}) = (C^\infty(\mathbf{R}^2,\mathbf{R}) \times \mathbf{R}^2)/{\sim},$$

the space of germs of real-valued functions on $\mathbf{R}^2$.

Then we have

Proposition 4.2 *For a germ $f : (\mathbf{R}^2, x_0) \to (\mathbf{R}, y_0)$, $f_{x_0} \in \mathcal{G}(\mathbf{R}^2,\mathbf{R})$, the following conditions are equivalent to each other:*

(1) *There exists an open neighbourhood V of $f : (\mathbf{R}^2, x_0) \to (\mathbf{R}, y_0)$ in $\mathcal{G}(\mathbf{R}^2,\mathbf{R})$ the quotient set $V/{\sim_\mathcal{A}}$ is a finite set.*

(2) *For the equivalence class $[f_{x_0}] \in \mathcal{G}(\mathbf{R}^2,\mathbf{R})/{\sim_\mathcal{A}}$ of f_{x_0} for the equivalence relation $\sim_\mathcal{A}$, there exists an open neighbourhood of $[f_{x_0}]$ in $\mathcal{G}(\mathbf{R}^2,\mathbf{R})/{\sim_\mathcal{A}}$ which consists of a finite number of points.*

If one of conditions is fulfilled, we call $f : (\mathbf{R}^2, x_0) \to (\mathbf{R}, y_0)$ *0-modal* (or *simple*), and also call the equivalence class $[f_{x_0}]$ *0-modal* (*simple*). In general, for a topological space $(\mathcal{G}, \mathcal{O})$ we can call a point $g \in \mathcal{G}$ *0-modal* (or *simple*) if there exists an open finite neighbourhood $U \in \mathcal{O}$ of g in $\mathcal{G}$. We denote by $\Sigma_{NS} \subset \mathcal{G}(\mathbf{R}^2,\mathbf{R})$ the set consisting of diffeomorphism classes of non-simple germs. Then $\mathcal{G}(\mathbf{R}^2,\mathbf{R}) \setminus \Sigma_{NS}$ designates the set consisting of diffeomorphism classes of simple germs. Then we have

Theorem 4.3 The quotient space $(\mathcal{G}(\mathbf{R}^2,\mathbf{R}) \setminus \Sigma_{NS})/{\sim_\mathcal{A}}$ is homeomorphic to the "ADE-space" (Figure 1).

For the proof, see [1].

Let X be a set and $\mathcal{W} = \{W_\mu\}$ is a family of subsets of X. Then there exists the minimal topology $\mathcal{O}$ containing $\mathcal{W}$. We call it the *topology generated by* $\mathcal{W}$.

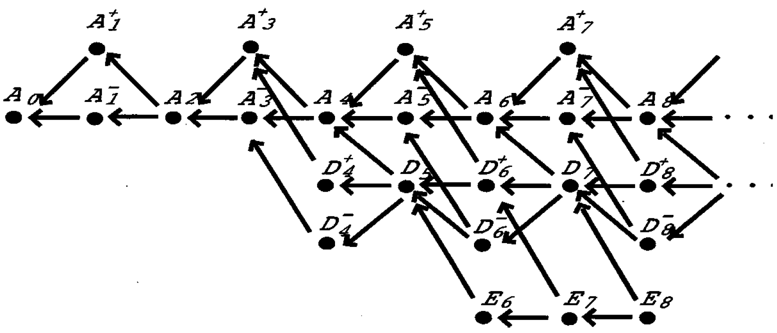

Figure 1: ADE-space

Let $X = (X, \mathcal{O}_X)$ be a topological space, and Y a subset of X. Then we collect subsets of Y of the form $Y \cap V$ for any open subset $V \in \mathcal{O}_X$ of X. Then we get a topological structure $\mathcal{O}_Y$, which is called the *relative topology* on Y.

Let $X = (X, \mathcal{O}_X)$ be a topological space, $\sim$ be an equivalence relation on X. The we give the *quotient topology* on $X/\!\sim$ by setting

$$\mathcal{O}_{X/\sim} := \{U \subseteq X/\!\sim \mid \pi^{-1}(U) \in \mathcal{O}_X\}.$$

Then $U \subseteq X/\!\sim$ is open if and only if $\pi^{-1}(U)$ is open in X.

Let X, Y be topological spaces. A mapping $f : X \to Y$ is called *continuous* if for any open subset U of Y, the inverse image $f^{-1}(U)$ is an open subset of X.

A mapping $\varphi : X \to Y$ is called a *homeomorphism* if φ is one-to one onto continuous mapping and the inverse mapping φ^{-1} is also continuous. If there is a homeomorphism from X to Y, then we call X and Y are *homeomorphic.*

Example 4.4 (1) Let $\sim$ be an equivalence relation on $\mathbf{R}$ defined by the condition that $x \sim x'$ if and only if $x' = x$ or $x' = -x$. We give on $\mathbf{R}/\!\sim$ the quotient topology from $\mathbf{R}$. On the other hand we give the relative topology on the half line $\mathbf{R}_{\geq 0} = \{x \in \mathbf{R} \mid x \geq 0\}$ from $\mathbf{R}$. The we see $\mathbf{R}/\!\sim$ and $\mathbf{R}_{\geq 0}$ are homeomorphic.

(2) We define another equivalence relation $\approx$ on $\mathbf{R}$ by that $x \approx x'$ if and only if $x = x' = 0$ or $xx' \neq 0$. The quotient set $\mathbf{R}/\!\approx$ consists of two equivalence relations: $\mathbf{R}/\!\approx\, = \{[0], [1]\}$. The quotient topology on $\mathbf{R}/\!\approx$ is given by $\{\emptyset, \{[1]\}, \{[0], [1]\}\}$. This topological space can be indicated by the diagram:

$$\bullet \longleftarrow \bullet$$

Let $(\Lambda, \leq)$ be a partially ordered set: A relation $v \leq v'$ on a set Λ is defined and satisfies that $v \leq v$, $(v \leq v', v' \leq v \Rightarrow v = v')$ and $(v \leq v', v' \leq v'' \Rightarrow v \leq v'')$. Then a subset $V \subseteq \Lambda$ is called *saturated* if, for any $v \in V, v' \in \Lambda$, $v' \leq v$ implies $v' \in V$. Then the family $\mathcal{O}_\Lambda$ of all saturated subsets of Λ satisfies the condition of topology. It is called the *topology induced from the ordering.*

5 Differentiable structure of mapping space quotients

In this section we introduce the method to give a differentiable structure on mapping spaces and their quotients. See also [18].

There are known several methods: For instance, the Eells' method based on Frechet differentials and Omori's method of "ILB manifold"(inverse limit Banach manifold). Our method is new and easy to apply compared with other known methods.

5.1 What are structures?

Let $\{X_\nu\}$ be a family of sets. The family X_ν is supposed to consist of quotients of subspaces of a topological space, in particular a mapping space $C^\infty(N, M)$ for manifolds N, M.

To define a "differentiable structure" on each X_ν from $\{X_\nu\}$, it is sufficient to give a criterion, for each pair $X_\nu, X_{\nu'}$, X_ν and $X_{\nu'}$ are "diffeomorphic". For that it is sufficient to give a criterion that a mapping $\Phi : X_\nu \to X_{\nu'}$ is "differentiable" or not.

Then, for example, how should we define that a given mapping $\Phi : C^\infty(N, M) \to C^\infty(L, W)$ (L, W are manifolds) is "differentiable"?

Let $\Phi : C^\infty(N, M) \to C^\infty(L, W)$ be a mapping. Then, for each differentiable mapping $f \in C^\infty(N, M)$, there corresponds a differentiable mapping $\Phi(f) \in C^\infty(L, W)$. Now we propose to call Φ differentiable if, for any "differentiable" family $h_\lambda \in C^\infty(N, M)$, $\Phi(h_\lambda) \in C^\infty(L, W)$ is "differentiable", where the "parameter" λ runs over a finite dimensional manifold Λ. In fact moreover we demand that Φ is continuous. As an ordinary term in global analysis and differential topology, we call $h_\lambda : N \to M$, $(\lambda \in \Lambda)$ is a *differentiable family* if there exists a differentiable mapping $H : \Lambda \times N \to M$ which satisfies $h_\lambda(x) = H(\lambda, x)$ for each $(\lambda, x) \in \Lambda \times N$. Then the mapping $h : \Lambda \to C^\infty(N, M)$ defined by $h(\lambda) = h_\lambda$ is called *differentiable* naturally.

Then for $\Phi(h_\lambda) \in C^\infty(L, W)$, we can take a differentiable mapping $G : \Lambda \times L \to W$ with $\Phi(h_\lambda)(x') = G(\lambda, x'), (\lambda, x') \in L \times W$. Therefore we can take the derivative of $\Phi(h_\lambda)$ with respect to λ.

5.2 Differentiability along finite dimensional directions.

Consider another example. How to define the differentiability of a functional $\Psi : C^\infty(L, W) \to \mathbf{R}$? The real value $\Psi(g)$ is determined for each mapping $g \in C^\infty(L, W)$. The function $\Psi(g_\lambda)$ of variable λ is determined for finite dimensional differentiable family $g_\lambda \in C^\infty(L, W), \lambda \in \Lambda$. Then we call a mapping $\Psi : C^\infty(L, W) \to \mathbf{R}$ *differentiable* if the function $\Psi(g_\lambda)$ is differentiable on λ. We regard each $g_\lambda \in C^\infty(L, W)$ as a point in the space $C^\infty(L, W)$. Then the family of mapping $g_\lambda \in C^\infty(L, W)$ is regarded as a finite dimensional subspace in $C^\infty(L, W)$. The family $\Psi(g_\lambda)$ is the restriction of Ψ to there, and we look at the differentiability of $\Psi(g_\lambda)$ in the ordinary sense. The differentiability we are going to define may be called the *differentiability along finite dimensional directions.*

If $\Phi : C^\infty(N, M) \to C^\infty(L, W)$ and $\Psi : C^\infty(L, W) \to \mathbf{R}$ are differentiable then the composition $\Psi \circ \Phi : C^\infty(N, M) \to \mathbf{R}$ is differentiable. If fact, for any differentiable family $h_\lambda \in C^\infty(N, M)$, we have $(\Psi \circ \Phi)(h_\lambda) = \Psi(\Phi(h_\lambda))$ and $\Phi : C^\infty(N, M) \to C^\infty(L, W)$ is differentiable, we see $\Phi(h_\lambda)$ is differentiable on λ. Since Ψ is differentiable, $\Psi(\Phi(h_\lambda))$ is differentiable, so is $(\Psi \circ \Phi)(h_\lambda)$ on λ.

We have defined that $\Psi : C^\infty(L, W) \to \mathbf{R}$ is differentiable. On the other hand, since $\mathbf{R}$ is identified with $C^\infty(\{\mathrm{pt}\}, \mathbf{R})$, we can regard $\Psi : C^\infty(L, W) \to C^\infty(\mathrm{pt}, \mathbf{R})$. Then Ψ is differentiable in the sense of the first definition. In fact, for any differentiable family $g_\lambda \in C^\infty(L, W)$, $\Psi(g_\lambda)$ is differentiable on λ. If we define $H : \Lambda \times \{\mathrm{pt}\} \to \mathbf{R}$ by $H(\lambda, \mathrm{pt}) = \Psi(g_\lambda)$, then H is differentiable. By definition, $\Psi : C^\infty(L, W) \to C^\infty(\{\mathrm{pt}\}, \mathbf{R})$ is differentiable.

Example 5.1 We define $\Psi : C^\infty(S^1, \mathbf{R}^2) \to \mathbf{R}$ by $\Psi(f) := \displaystyle\int_{S^1} f^*(xdy)$ where x, y is the system of coordinates on $\mathbf{R}^2$. Then Ψ is differentiable.

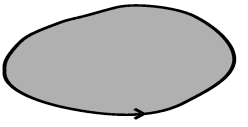

Figure 2: The area surrounded by a plane curve.

5.3 Differential structure of manifold quotients.

First we start with the case that the mapping space is a subset of a finite dimensional manifold N which will be identified with the space $C^\infty(\{\mathrm{pt}\}, N)$.

Let N be a differentiable manifold, S a subset of N, and $\sim$ a equivalence relation on S. Assume Λ, M and Q are also differentiable manifolds which play a role of "test space".

Then the differentiability is introduced inductively as follows:

(1) We call a mapping $h : \Lambda \to S$ from a manifold to a subset of a manifold *differentiable* if the composed mapping $h : \Lambda \to S \hookrightarrow N$ is a differentiable mapping from the manifold Λ to the manifold N.

(2) We call a mapping $k : S \to Q$ from a subset of a manifold to a manifold *differentiable* if k is continuos, and for any differentiable mapping $h : \Lambda \to S$ in the sense of (1), the composed mapping $k \circ h : \Lambda \to Q$ is a differentiable mapping from the manifold Λ to the manifold Q.

(3) We call a mapping $\ell : S/\!\sim \to Q$ from a quotient of a subset of a manifold to a manifold *differentiable* if the composed mapping $\ell \circ \pi : S \to S/\!\sim \to Q$ is differentiable in the sense of (2).

(4) We call a mapping $m : \Lambda \to S/\!\sim$ from a manifold to a quotient of a subset of a manifold *differentiable* if, for any differentiable mapping $\ell : S/\!\sim \to Q$ in the sense of (3), the composed mapping $\ell \circ m : \Lambda \to Q$ is a differentiable mapping from the manifold Λ to the manifold Q.

More generally:

(5) We call a mapping $\varphi : S/\!\sim \to T/\!\approx \leftarrow T \subseteq M$ from a quotient of a subset of a manifold to a subset of a manifold *differentiable* if φ is continuous and, for any differentiable mapping $\ell : T/\!\approx \to Q$ in the sense of (3), the composed mapping $\ell \circ \varphi : S/\!\sim \to Q$ is differentiable in the sense of (3).

(6) A mapping $\varphi : S/\!\sim \to T/\!\approx$ is called a *diffeomorphism* if φ is differentiable in the sense of (5), bijective, and the inverse mapping $\varphi^{-1} : T/\!\approx \to S/\!\sim$ is differentiable in the sense of (5).

(7) The quotient spaces $S/\!\sim$ and $T/\!\approx$ are called *diffeomorphic* if there exists a diffeomorphism $\varphi : S/\!\sim \to T/\!\approx$.

Remark 5.2 There is a different definition for the stage (2) (cf. [24]): A mapping $k : S \to Q$ is called differentiable if there exists an open neighbourhood U in N and a differentiable mapping $\bar{k} : U \to Q$ satisfying $\bar{k}|_S = k$. Compared with this definition which is based on extensions of mappings on S, our definition is based on parametrisations of S and may be called a "parametric-minded" definition.

Example 5.3 (Differentiable structure on orbifolds). Let G be a finite subgroup of $\mathrm{GL}(n, \mathbf{R})$ which acts on $\mathbf{R}^n$ naturally.

By the above general theory, we can endow with the " orbifold" $\mathbf{R}^n/G$ the ordinary differentiable structure.

Example 5.4 The quotient space $\mathbf{R}/\!\sim$ is diffeomorphic to $\mathbf{R}_{\geq 0}$, where $\sim$ is an equivalence relation on $\mathbf{R}$ defined by that $x \sim x'$ if and only if $x' = \pm x$.

In fact $\varphi : \mathbf{R}/\!\sim\, \to \mathbf{R}_{\geq 0}$, $\varphi([x]) = x^2$ is a diffeomorphism. For, $\varphi \circ \pi : \mathbf{R} \to \mathbf{R}_{\geq 0}$, $(\varphi \circ \pi)(x) = x^2$ is a continuous differentiable mapping by (1), we see φ is a differentiable mapping by (3). The inverse mapping is given by $\psi : \mathbf{R}_{\geq 0} \to \mathbf{R}/\!\sim$, $\psi(y) = [\sqrt{y}]$. To see ψ is differentiable, we check, based on (5), for any differentiable mapping $\ell : \mathbf{R}/\!\sim\, \to Q$, that $\ell \circ \psi : \mathbf{R}_{\geq 0} \to Q$ is differentiable. By (3), $\ell \circ \pi : \mathbf{R} \to Q$ is differentiable. Since $(\ell \circ \pi)(x) = (\ell \circ \pi)(-x)$, we see there exists a differentiable mapping $\rho : \mathbf{R} \to Q$ with $(\ell \circ \pi)(x) = \rho(x^2)$. Then $(\ell \circ \psi)(y) = \ell([\sqrt{y}]) = (\ell \circ \pi)(\sqrt{y}) = \rho(y)$. Thus $\ell \circ \psi$ is differentiable. $\square$

Example 5.5 We give the equivalence relation $\sim$ on $\mathbf{R}^2$ by that $(x, y) \sim (x', y')$ if and only if $(x', y') = \pm(x, y)$. Then we see $\mathbf{R}^2/\!\sim$ is homeomorphic to $\mathbf{R}^2$ but $\mathbf{R}^2/\!\sim$ is not diffeomorphic to $\mathbf{R}^2$.

The mapping $s : \mathbf{R}^2/\!\sim\, \to \mathbf{R}^2$, $s([(x, y)]) = (x^2 - y^2, 2xy)$ is a homeomorphism. However s is not a diffeomorphism. Moreover we see that there exists no diffeomorphism between $\mathbf{R}^2/\!\sim$ and $\mathbf{R}^2$. To see that, suppose that there exist a differentiable mapping $\psi : \mathbf{R}^2 \to \mathbf{R}^2/\!\sim$ and a differentiable mapping $\varphi : \mathbf{R}^2/\!\sim\, \to \mathbf{R}^2$ satisfying $\psi \circ \varphi = \mathrm{id}, \varphi \circ \psi = \mathrm{id}$. Since $\varphi \circ \pi : \mathbf{R}^2 \to \mathbf{R}^2$ is differentiable and invariant under the transformation $(x, y) \mapsto (-x, -y)$, there exists a differentiable mapping $\rho : \mathbf{R}^3 \to \mathbf{R}^2$ satisfying $(\varphi \circ \pi)(x, y) = \rho(x^2, xy, y^2)$. Therefore there exists a differentiable mapping $\Phi : \mathbf{R}^2/\!\sim\, \to \mathbf{R}^3$ with $(\Phi \circ \pi)(x, y) = (x^2, xy, y^2)$ so with $\varphi \circ \pi = \rho \circ \Phi \circ \pi$. Since π is a surjective, we have $\varphi = \rho \circ \Phi$. Therefore $\mathrm{id} = \varphi \circ \psi = \rho \circ (\Phi \circ \psi) : \mathbf{R}^2 \to \mathbf{R}^2$. However the image of $\Psi := \Phi \circ \psi : \mathbf{R}^2 \to \mathbf{R}^3$ is contained in $\{(x^2, xy, y^2) \mid (x, y) \in \mathbf{R}^2\} = \{(X, Y, Z) \in \mathbf{R}^3 \mid XZ - Y^2 = 0\}$ and thus $\mathrm{rank}_0 \Psi \leq 1$. This leads a contradiction. $\square$

Example 5.6 (The differentiable structure of a quotient space by the complex conjugation) Let $\mathrm{conj} : \mathbf{C}^n \to \mathbf{C}^n$, $\mathrm{conj}(z) = \bar{z}$ be the complex conjugation. The quotient space $\mathbf{C}/\mathrm{conj}$ is diffeomorphic to $\mathbf{R} \times \mathbf{R}_{\geq 0}$. For $\mathbf{C}^2/\mathrm{conj}$, it is homeomorphic to $\mathbf{R}^4$ and it is not diffeomorphic to $\mathbf{R}^4$. Then we endow $\mathbf{C}^2/\mathrm{conj}$ with the differentiable structure induced from the standard one on $\mathbf{R}^4$. If we endow a differentiable structure on $\mathbf{C}P^2/\mathrm{conj}$ in the same way as above, we see that $\mathbf{C}P^2/\mathrm{conj}$ is diffeomorphic to the 4-sphere S^4. (The theorem of Kuiper, Massey, Arnold.)

5.4 Differentiable structures on mapping space quotients.

Let N, M, L, P be differentiable manifolds. Moreover, in this section, Λ, Q always designate (finite dimensional) differentiable manifolds respectively.

Let $X \subseteq C^\infty(N, M)$ be a subset. Then, such a set X is a *mapping space.*

(1) We call a mapping $h : \Lambda \to X$ *differentiable* if there exists a differentiable mapping (between manifolds) $H : \Lambda \times N \to M$ satisfying $H(\lambda, x) = h(\lambda)(x) \in M$, $(\lambda \in \Lambda, x \in N)$.

(2) We call a mapping $k : X \to Q$ *differentiable* if

k is a continuous mapping and, for any for any differentiable mapping $h : \Lambda \to X$ in the sense of (1), the composition $k \circ h : \Lambda \to Q$ is a differentiable mapping between manifolds.

Now, if $\sim$ is an equivalence relation on a mapping space X, then we get the quotient space $X/\sim$. Such a quotient space $X/\sim$ is called a *mappings space quotient.* Then the projection $\pi : X \to X/\sim$ is defined by $\pi(x) = [x]$ (the equivalence class of x).

(3) We call a mapping $\ell : X/\sim \to Q$ *differentiable* if the composition $\ell \circ \pi : X \to Q$ with the projection π is differentiable in the sense of (2).

(4) We call a mapping $m : \Lambda \to X/\sim$ *differentiable* if, for any differentiable mapping $\ell : X/\sim \to Q$ in the sense of (3), the composition $\ell \circ m : \Lambda \to Q$ is a differentiable mapping between manifolds.

Lemma 5.7 *If $h : \Lambda \to X$ is differentiable in the sense of (1), $\pi \circ h : \Lambda \to X/\sim$ is differentiable in the sense of (4).*

Proof: For any differentiable mapping $\ell : X/\sim \to Q$ in the sense of (3), the composition $\ell \circ \pi : X \to Q$ differentiable in the sense of (2). Therefore $(\ell \circ \pi) \circ h = \ell \circ (\pi \circ h) : \Lambda \to Q$ is differentiable. Hence $\pi \circ h$ differentiable in the sense of (4). □

(5) In general, we call a mapping $\varphi : X/\sim \to Y/\approx$ from a mapping space quotient $X/\sim$ to another mapping space quotient $Y/\approx \leftarrow Y \subseteq C^\infty(L, P)$ *differentiable* if φ is a continuous mapping and, for any differentiable mapping $\ell : Y/\approx \to Q$ in the sense of (3), the composition $\ell \circ \varphi : X/\sim \to Q$ is differentiable in the sense of (3).

(6) Then we call a mapping $\varphi : X/\sim \to Y/\approx$ a *diffeomorphism* if φ is differentiable in the sense of (5), φ is a bijection and the inverse mapping $\varphi^{-1} : Y/\approx \to X/\sim$ is also differentiable in the sense of (5).

(7) Then we call two mapping space quotients $X/\sim$ and $Y/\approx$ *diffeomorphic* if there exists a diffeomorphism $\varphi : X/\sim \to Y/\approx$ in the sense of (6).

Lemma 5.8 *The following two conditions are equivalent to each other:*

(i) $\varphi : X/\sim \to Y/\approx$ is differentiable in the sense of (5).

(ii) $\varphi : X/\sim \to Y/\approx$ is a continuous mapping and, for any differentiable mapping $m : \Lambda \to X/\sim$ in the sense of (4), $\varphi \circ m : \Lambda \to Y/\approx$ differentiable in the sense of (4).

Proof: (i) $\Rightarrow$ (ii): Let $\ell : Y/\approx \to Q$ be a differentiable mapping in the sense of (3). By (i), $\ell \circ \varphi : X/\sim \to Q$ differentiable in the sense of (3). Then $(\ell \circ \varphi) \circ m = \ell \circ (\varphi \circ m) : \Lambda \to Q$ is a differentiable mapping. Therefore $\varphi \circ m : \Lambda \to Y/\approx$ is a differentiable mapping in the sense of (4).

(ii) $\Rightarrow$ (i): For any differentiable mapping $\ell : Y/\approx \to Q$ in the sense of (3), we check that $\ell \circ \varphi : X/\sim \to Q$ is differentiable in the sense of (3), namely that $(\ell \circ \varphi) \circ \pi : X \to Q$ is differentiable in the sense of (2). Then we check, for any differentiable $h : \Lambda \to X$ in the sense of (1), that $((\ell \circ \varphi) \circ \pi) \circ h : \Lambda \to Q$ is differentiable. By 5.7, $\pi \circ h : \Lambda \to X/\sim$ is differentiable in the sense of (4). Therefore, by (ii), $(\varphi \circ \pi) \circ h = \varphi \circ (\pi \circ h) : \Lambda \to Y/\approx$ is differentiable in the sense of (4). Hence $\ell \circ (\varphi \circ \pi) \circ h = ((\ell \circ \varphi) \circ \pi) \circ h : \Lambda \to Q$ is is a differentiable mapping. Thus φ is differentiable in the sense of (5). □

Lemma 5.9 *For C^∞ topology on $C^\infty(N, M)$, a differentiable mapping $h : \Lambda \to X \subseteq C^\infty(N, M)$ in the sense of (1) is a continuous mapping.*

Proof: By the assumption, there exists a differentiable mapping $H : \Lambda \times N \to M$ which satisfies $H(\lambda, x) = h(\lambda)(x)$. Take an open subset of $C^\infty(N, M)$ of the form $W(r, K, U)$, where $K \subseteq N$ is a compact subset and $U \subseteq J^r(N, M)$ is an open subset.

Suppose, for a $\lambda_0 \in \Lambda$, $h(\lambda_0) = H|_{\lambda_0 \times N} : N \times M$ belongs to $W(r, K.U)$. Define $j_1^r H : \Lambda \times N \to J^r(N, M)$ by $j_1^r H(\lambda, x) = j^r(H|_{\lambda \times N})(x)$. Then $j_1^r H$ is a differentiable mapping in the ordinary sense. In particular it is continuous. From the assumption $h(\lambda_0) \in W(r, K, U)$, $(j_1^r H)^{-1}(W(r, K, U))$ is an open neighbourhood of $\lambda_0 \times K$. Since K is compact, there exists an open neighbourhood V of λ_0 such that $V \times K \subseteq (j_1^r H)^{-1}(W(r, K, U))$. This means that $\lambda_0 \in V \subseteq h^{-1}(W(r, K, U))$. Therefore $h^{-1}(W(r, K, U))$ is open. Noting that $h^{-1}(W(r, K, U) \cap W(r', K', U')) = h^{-1}(W(r, K, U)) \cap h^{-1}(W(r', K', U'))$, $h^{-1}(\cup W_\nu) = \cup h^{-1}(W_\nu)$, we see h is continuous. □

Remark 5.10 Lemma 5.9 does not hold for Whitney C^∞ topology. For example, in $X = C^\infty(\mathbf{R}, \mathbf{R})$, consider the differentiable mapping $h : \mathbf{R} \to C^\infty(\mathbf{R}, \mathbf{R})$ defined by the differentiable mapping $H(\lambda, x) := \lambda$. Then $h(0)$ is identically 0. Its graph is $\mathbf{R} \times 0 \subset \mathbf{R} \times \mathbf{R}$. Then there exists an open set U containing $\mathbf{R} \times 0$ such that $h^{-1}(W(U)) = \{0\}$. Then $W(U)$ is an open subset of $C^\infty(\mathbf{R}, \mathbf{R})$ with respect to Whitney C^∞ topology, while $h^{-1}(W(U)) = \{0\} \subset \mathbf{R}$ is not open in $\mathbf{R}$. Therefore h is not continuous in Whitney C^∞ topology.

Remark 5.11 In the above definition (2), the continuity of h is not implied from just the condition that for any differentiable mapping $h : \Lambda \to X$ in the sense (1), the composition $k \circ h : \Lambda \to Q$ is differentiable.

In fact set $X = \{1/n\} \cup \{0\} \subset \mathbf{R} = C^\infty(\{\mathrm{pt}\}, \mathbf{R})$ and $Y = \{0, 1\} = C^\infty(\{\mathrm{pt}\}, \{0, 1\})$.

Define $k : X \to Y$ by $k(1/n) = 1, k(0) = 0$. Then any differentiable mapping $h : \Lambda \to X$ is locally constant, and so is $k \circ h : \Lambda \to Y$. Then $k \circ h$ is differentiable, while k is not continuous.

Thus, in the definition (2), we need the continuity of k.

5.5 Properties.

Lemma 5.12 *Let M be a differentiable manifold. Then M is diffeomorphic to $C^\infty(\{\mathrm{pt}\}, M)$.*

Proof: Both $\varphi : M \to C^\infty(\{\mathrm{pt}\}, M)$, $\varphi(x)(\mathrm{pt}) := x$, and $\psi : C^\infty(\{\mathrm{pt}\}, M) \to M$, $\psi(f) := f(\mathrm{pt})$, are differentiable and inverse mappings to each other. □

Lemma 5.13 *(1) The identity mapping* $\mathrm{id} : X/\!\sim\, \to X/\!\sim$ *is differentiable.*

(2) If $f : X/\!\sim\, \to Y/\!\approx$ and $g : Y/\!\approx\, \to Z/\!\equiv$ are both differentiable, then the composition $g \circ f : X/\!\sim\, \to Z/\!\equiv$ is differentiable.

Proof: (1) is clear. (2) Since f and g are continuous, $g \circ f$ is continuous. Let $m : \Lambda \to X/\!\sim$ be any differentiable mapping. Then, from the assumption, $f \circ m : \Lambda \to Y/\!\approx$ is a differentiable mapping. Moreover we see $g \circ (f \circ m) = (g \circ f) \circ m : \Lambda \to Z/\!\equiv$ is differentiable. Therefore $g \circ f$ is differentiable. □

Lemma 5.14 *(1) The quotient mapping $\pi : X \to X/\!\sim$ is differentiable. (2) A mapping $f : X/\!\sim\, \to Y/\!\approx$ is differentiable if and only if $f \circ \pi : X \to Y/\!\approx$ is differentiable.*

Proof: (1) That π is continuous is clear. For any differentiable mapping $\ell : X/\sim \to Q$, $\ell \circ \pi : X \to Q$ is differentiable. Therefore π is differentiable. (2) f is continuous if and only if $f \circ \pi$ is continuous. If f is differentiable, then $f \circ \pi$ is differentiable. Conversely, assume $f \circ \pi$ is differentiable and take any differentiable mapping $\ell : Y/\approx \to Q$. $\ell \circ (f \circ \pi) = (\ell \circ f) \circ \pi : X \to Q$ is differentiable. Hence $\ell \circ f$ is differentiable. Thus f is differentiable. □

Lemma 5.15 *If N and N' are diffeomorphic, and, M and M' are diffeomorphic, then $C^\infty(N, M)$ and $C^\infty(N', M')$ are diffeomorphic.*

Example 5.16 Let $K \subset \mathbf{R}$ be a compact subset. Define $I : C^\infty(\mathbf{R}, \mathbf{R}) \to \mathbf{R}$ by $I(f) = \int_K f(x)dx$. Then the mapping I is differentiable (in the sense of the definition (2)).

5.6 Product space.

For $X \subseteq C^\infty(N, M), Y \subseteq C^\infty(L, P)$, we set

$$X \times Y = \{F \in C^\infty(N \amalg L, M \amalg P) \mid F(N) \subseteq M, F(L) \subseteq P\},$$

where $N \amalg L$ is the disjoint union of N and L. Then identify $X/\sim \times Y/\approx$ with $(X \times Y)/\equiv$, where we define $(f, g) \equiv (f', g')$ by that $f \sim f'$ and $g \approx g'$. Thus the product of mapping space quotients is regarded as a mapping space quotient.

Example 5.17 The mapping $C^\infty(N, M) \times N \to M$ defined by $(f, x) \mapsto f(x)$ is a differentiable mapping. This means that if $f_n \to f, x_n \to x$ then $f_n(x_n) \to f(x)$, $(n \to \infty)$ and moreover that $f(x)$ is differentiable both for f and x.

Lemma 5.18 *The mapping $\Phi : C^\infty(N, M) \times C^\infty(M, L) \to C^\infty(N, L)$ defined by $\Phi(f, g) = g \circ f$ is a differentiable mapping.*

Proof: By proposition 3.3, Φ is continuous. Suppose $h : \Lambda \to C^\infty(N, M) \times C^\infty(M, L)$ is a differentiable mapping. Suppose a differentiable mapping $H : \Lambda \times (N \amalg M) \to M \amalg L$ defined the differentiable mapping h. Then we have $H(\Lambda \times N) \subseteq M, H(\Lambda \times M) \subseteq L$. Set $F := H|_{\Lambda \times N}$ and $G := H|_{\Lambda \times M}$. Then $f_\lambda(x) = F(\lambda, x)$ and $g_\lambda(f_\lambda(x)) = g_\lambda(F(x, \lambda) = G(\lambda, F(\lambda, x))$ is differentiable. □

6 Examples

We give several examples as the applications of our theory.

6.1 The space of triangles.

Let $N = \{\alpha, \beta, \gamma\}$ be a 0-dimensional manifold consisting of three points, endowed with the discrete topology. Set $M = \mathbf{R}^2$. The mapping space $C^\infty(N, M)$ is diffeomorphic to $\mathbf{R}^6$ by the correspondence $f \mapsto (f(\alpha), f(\beta), f(\gamma))$. Set

$$\mathcal{T} := \{f \in C^\infty(N, M) \mid f(\alpha), f(\beta), f(\gamma) \text{ is not collinear}\}.$$

Then $\mathcal{T}$ is the "space of triangles", which is identified with the open subset of $\mathbf{R}^6$

$$\left\{ (a_1, a_2; b_1, b_2; c_1, c_2) \in \mathbf{R}^6 \;\middle|\; \begin{vmatrix} 1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \neq 0 \right\}.$$

Now we are going to classify triangles by congruences. We set on $\mathbf{R}^2$ the Euclidean metric. Then we denote the group of motions on $\mathbf{R}^2$ by Euclid($\mathbf{R}^2$):

$$\mathrm{Euclid}(\mathbf{R}^2) := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ P & a & b \\ Q & c & d \end{pmatrix} \;\middle|\; (P, Q) \in \mathbf{R}^2, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{is a orthogonal matrix} \right\}.$$

Here we assume the reflections are contained in Euclid($\mathbf{R}^2$).

For $A = (x_1, x_2) \in R^2$,

$$\begin{pmatrix} 1 & 0 & 0 \\ P & a & b \\ Q & c & d \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ ax_1 + bx_2 + P \\ cx_1 + dx_2 + Q \end{pmatrix}.$$

Namely $g(A) = g(x_1, x_2) = (ax_1 + bx_2 + P, cx_1 + dx_2 + Q)$. Now set $G = S_3 \times \mathrm{Euclid}(\mathbf{R}^2)$, where S_3 is the symmetry group of order three, which is equal to Diff(N) in this case. We define the action of G on $C^\infty(N, M) \cong \mathbf{R}^6$ by

$$(\sigma, g)(A, B, C) = \sigma(g(A), g(B), g(C))$$

the permutation by σ, for $(A, B, C) \in \mathbf{R}^6$. Then $\mathcal{T} \subset C^\infty(N, M)$ is G-invariant. The quotient space $\mathcal{T}/G$ is the space of congruence classes of triangles.

Theorem 6.1 *The space of $\mathcal{T}/G$ of congruence classes of triangles is diffeomorphic to $\mathbf{R}_{>0} \times C$, where*

$$C := \{(x, y) \in \mathbf{R}^2 \mid x^2 - y^3 \leq 0\}$$

is the narrower domain defined by the cusp curve. In particular the space of $\mathcal{T}/G$ of congruence classes of triangles is homeomorphic to $\mathbf{R}^2 \times \mathbf{R}_{\geq 0} = \{(x, y, z) \mid z \geq 0\}$.

The part $\mathbf{R}_{>0}$ from $\mathbf{R}_{>0} \times C$ represents the parameter of similarity. The summit of C corresponds to the congruence classes of equilateral triangles, while the edge corresponds to the congruence classes of isosceles triangles.

Proof of Theorem 6.1:

The mapping

$$\mathbf{R}^6 \supset \mathcal{T} \ni (A, B, C) \mapsto (BC, CA, AB) \in (\mathbf{R}_{>0})^3$$

induces a mapping $\Phi_1 : \mathcal{T}/G \to (\mathbf{R}_{>0})^3/S_3$. Define $V : (\mathbf{R}_{>0})^3 \to (\mathbf{R}_{>0})^3$ by $V(a, b, c) := (a + b + c, ab + bc + ca, abc)$. Then V induces $\bar{V} : (\mathbf{R}_{>0})^3/S_3 \to (\mathbf{R}_{>0})^3$. Let X be the image of $\bar{V}$. Then we see that $\Phi := \bar{V} \circ \Phi_1 : \mathcal{T}/G \to X$ is a diffeomorphism. Moreover we see that X is diffeomorphic to $\mathbf{R}_{>0} \times C$. $\square$

6.2 Diffeomorphism groups.

Let N be a compact manifold without boundary. Then the group $\mathrm{Diff}^\infty(N)$ of diffeomorphisms on N is a topological group in the C^∞ topology. Moreover the composition $m : \mathrm{Diff}^\infty(N) \times \mathrm{Diff}^\infty(N) \to \mathrm{Diff}^\infty(N), m(\varphi, \psi) = \psi \circ \varphi$ and the inverse $i : \mathrm{Diff}^\infty(N) \to \mathrm{Diff}^\infty(N), i(\varphi) = \varphi^{-1}$ are differentiable. Thus, in this sense, $\mathrm{Diff}^\infty(N)$ is regarded as an infinite dimensional Lie group.

6.3 The space of knots.

The space $\mathrm{Emb}(S^1, \mathbf{R}^3) \subset C^\infty(S^1, \mathbf{R}^3)$ is an open subset. Set $G := \mathrm{Diff}^\infty(S^1) \times \mathrm{Diff}^\infty(\mathbf{R}^3)$. Then the group G acts on $C^\infty(S^1, \mathbf{R}^3)$ by $(\sigma, \tau)f := \tau \circ f \circ \sigma^{-1}$. Then $\mathrm{Emb}(S^1, \mathbf{R}^3)$ is G-invariant. Thus G acts also on $\mathrm{Emb}(S^1, \mathbf{R}^3)$.

Proposition 6.2 *The quotient space* $\mathrm{Emb}(S^1, \mathbf{R}^3)/G$ *is diffeomorphic to the countably infinite discrete space.*

We might claim that all knots are equal, from the viewpoint of differentiable structure; we need a global and concrete condition to define the "trivial knot"such as being the boundary of an embedded disk.

If we take the space $\mathrm{Imm}(S^1, \mathbf{R}^3)$ of immersions, instead of the space of embeddings, then the quotient space $\mathrm{Imm}(S^1, \mathbf{R}^3)/G$ is not a discrete space. The Vassiliev invariant ([28][29]) of knots can be understood as an invariant constructed from the embedding of the discrete space $\mathrm{Emb}(S^1, \mathbf{R}^3)/G$ into the non-discrete space $\mathrm{Imm}(S^1, \mathbf{R}^3)/G$.

By the same idea, consider the open subset

$$\mathrm{Gen}(S^1, \mathbf{R}^2) := \{f \in C^\infty(S^1, \mathbf{R}^2) \mid f \text{is generic}\}$$

of the space $C^\infty(S^1, \mathbf{R}^2)$ of parametric plane curves. Here a plane curve $f : S^1 \to \mathbf{R}^2$ is called *generic* if f is an immersion and its self-intersections are only transversal intersections. Then the group $G := \mathrm{Diff}^\infty(S^1) \times \mathrm{Diff}^\infty(\mathbf{R}^2)$ acts on $C^\infty(S^1, \mathbf{R}^2)$ and $\mathrm{Gen}(S^1, \mathbf{R}^2)$ is G-invariant.

Proposition 6.3 *The quotient space* $\mathrm{Gen}(S^1, \mathbf{R}^2)/G$ *is diffeomorphic to the countably infinite discrete space.*

Contrarily $\mathrm{Imm}(S^1, \mathbf{R}^2)/G$ is not discrete. The Arnol'd invariant [2] can be understood an invariant constructed by means of the embedding of the discrete space $\mathrm{Gen}(S^1, \mathbf{R}^2)/G$ into the non-discrete space $\mathrm{Imm}(S^1, \mathbf{R}^2)/G$.

6.4 Superspace of Riemannian structures.

Let N be a differentiable manifold of dimension n. The space

$$\mathcal{R}_N := \{\text{the Riemannian metrics on } N\}$$

can be regarded as a mapping space. In fact, a Riemannian metric on N determines, by definition, to each point $x \in N$ a positive definite symmetric bilinear form $g_x : T_xN \times T_xN \to \mathbf{R}$ depending on x in a differentiable way. It is given by a differentiable section $g : N \to T^*N \odot T^*N$ which possesses the positivity. Thus we can regard as $\mathcal{R}_N \subset C^\infty(N, T^*N \odot T^*N)$, where $T^*N \odot T^*N$ means the tensor product of cotangent bundle T^*N over N.

Now the diffeomorphism group Diff(N) acts on $\mathcal{R}_N$ naturally. The quotient space (orbit space)

$$\mathcal{S}_N := \mathcal{R}_N/\mathrm{Diff}(N)$$

is the space of isometry classes of Riemannian structures on N and is called the *super space* of N ([12][6]).

The projection $\pi : \mathcal{R}_N \to \mathcal{S}_N$ is differentiable. For each isometry class $[g] \in \mathcal{S}_N$, the fibre $\pi^{-1}([g])$ is diffeomorphic to the quotient space Diff(N)/Isom(N, g) of Diff(N) by the isometry group Isom(N, g) of (N, g).

Theorem 6.4 *Let $N = S^1$. Then $\mathcal{S}_{S^1}$ is diffeomorphic to $\mathbf{R}_{>0}$ and therefore to $\mathbf{R}$. The space $\mathbf{R}_{>0}$ corresponds to the total lengths of the one-dimensional Riemannian manifolds (S^1, g).*

One of interesting problem is to determine the superspace $\mathcal{S}_N$ for a general N. In the case $\dim N \geq 2$, the space $\mathcal{S}_N$ turns out to be infinite dimensional. For example, in the case $\dim N = 2$, the Gaussian curvature $K : N \to \mathbf{R}$ on N gives a functional moduli on $\mathcal{S}_N$.

By the way, among the Riemannian metrics on $N = S^2$, the round sphere, namely, the round sphere in the Euclidean $\mathbf{R}^3$ seems to be distinguished. Then we are led to the following conjecture:

For the standard metric g_0 on S^2, if $(\mathcal{S}_{S^2}, [g_0])$ and another $(\mathcal{S}_{S^2}, [g])$ are locally diffeomorphic, then $[g_0] = [g]$, namely g is isometric to the standard metric g_0.

6.5 The symplectic moduli space of plane curves.

Let $\mathrm{Emb}(S^1, \mathbf{R}^2) \subset C^\infty(S^1, \mathbf{R}^2)$ be the space of differentiable simple closed curves on the plane.

Let Symp($\mathbf{R}^2$) be the group of diffeomorphisms preserving the standard symplectic structure $\omega_0 = dx \wedge dy$.

The group $H = \mathrm{Diff}(S^1) \times \mathrm{Symp}(\mathbf{R}^2)$ acts on the space $\mathrm{Emb}(S^1, \mathbf{R}^2)$ in the natural way, namely, by $(\sigma, \tau)f := \tau \circ f \circ \sigma^{-1}$. The quotient space $\mathrm{Emb}(S^1, \mathbf{R}^2)/H$ is regarded as the space of symplectomorphism classes of simple closed curves.

Proposition 6.5 *The quotient space $\mathrm{Emb}(S^1, \mathbf{R}^2)/H$ is diffeomorphic to $\mathbf{R}_{\geq 0}$. The space $\mathbf{R}_{>0}$ corresponds to the area surrounded by curves.*

We consider again the open subset in $C^\infty(S^1, \mathbf{R}^2)$:

$$\mathrm{Gen}(S^1, \mathbf{R}^2) := \{f \in C^\infty(S^1, \mathbf{R}^2) \mid f \text{ is generic}\}.$$

The group $H = \mathrm{Diff}(S^1) \times \mathrm{Symp}(\mathbf{R}^2)$ acts also on $\mathrm{Gen}(S^1, \mathbf{R}^2)$.

Also another group $G = \mathrm{Diff}(S^1) \times \mathrm{Diff}^+(\mathbf{R}^2)$ acts on $\mathrm{Gen}(S^1, \mathbf{R}^2)$, where $\mathrm{Diff}^+(\mathbf{R}^2)$ is the group consisting of orientation preserving diffeomorphisms on $\mathbf{R}^2$.

For an $f \in \mathrm{Gen}(S^1, \mathbf{R}^2)$ consider the G-orbit $G \cdot f$ through f. The group acts on $G \cdot f$. The image $f(S^1)$ of a generic mapping $f : S^1 \to \mathbf{R}^2$ divides $\mathbf{R}^2$ into several regions. We put labels on bounded regions (Figure 3). Then for each $f' \in G \cdot f$, the bounded regions divided by $g'(S\ 1)$ have induced labels.

Then we define, for $f', f'' \in G \cdot f$, $f' \sim f''$ if there exists $(\sigma, \tau) \in H$ such that $f'' = \tau \circ f' \circ \sigma^{-1}$ and that τ preserves the labellings.

We call the quotient space $\mathcal{M}(f) := (G \cdot f)/\sim$ the *symplectic moduli space* of the plane curve f.

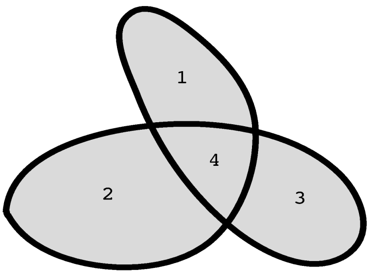

Figure 3: Labelling

Proposition 6.6 *The symplectic moduli space $\mathcal{M}(f) = (G \cdot f)/\sim$ of a generic plane curve f is diffeomorphic to $(\mathbf{R}_{>0})^r$, therefore it is diffeomorphic to $\mathbf{R}^r$, where r is the number of bounded regions surrounded by $f(S^1)$.*

The space $(\mathbf{R}_{>0})^r$ corresponds to areas of bounded domains. The structure of symplectic moduli spaces for more singular curves is studied in [18].

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